Lecture 1 $\qquad L^p$ outer measure theory. $\qquad$ 11/05/2015.

X metric space; All sets mentioned in Lecture are Borel-measurable and functions h are in

$\mathcal{B}(X) := \{$Borel meas. functions$\}$.

(1) Def^n. A premeasure is a function $\sigma : \mathbb{E} \to [0,\infty)$ defined on a collection of subsets of X. $(\mathbb{E} \subset 2^X)$.

σ generates an outer-measure on 2^X by:

$$\mu(E) := \inf_{\mathbb{E}'} \left\{ \sum_{E' \in \mathbb{E}'} \sigma(E') : \mathbb{E}' \subset \mathbb{E} \text{ s.t. } E \subset \bigcup_{E' \in \mathbb{E}'} E' \right\}.$$

Rmk.
- It's imperative that $\sigma \neq \infty$, because otherwise μ will be trivial.
- If $E \not\subset \bigcup_{E' \in \mathbb{E}'} E'$, then simply let $\mu(E) = +\infty$.

For any σ, μ will be an outer measure:

(I). $E \subset E' \Rightarrow \mu(E) \leq \mu(E')$.

(II). $\mu(\emptyset) = 0$.

(III). $\{E_i\}$ countable $\Rightarrow \mu(\bigcup_i E_i) \leq \sum_i \mu(E_i)$.

Pf. $E_i = \bigcup_j F_i^j$, s.t.

$$\cancel{\mu(E_i) \leq} \sum_j \sigma(F_i^j) \leq \mu(E_i) + 2^{-i}\varepsilon .$$

$$\Rightarrow \bigcup_i E_i \subset \bigcup_{i,j} F_i^j$$

$$\mu(\cup E_i) \leq \sum_{ij} \sigma(F_j^i) \leq \sum_i \mu(E_i) + \sum_i 2^{-i}\varepsilon .$$

$$\leq \sum_i \mu(E_i) + \varepsilon .$$

2. Defn. 2.3: A size is a map $S : B(X) \times \mathbb{E} \to [0,\infty]$.

Satisfying:

(I). $|f| \leq |g| \Rightarrow S(f,E) \leq S(g,E)$.

(II). $\lambda \in \mathbb{C} \Rightarrow S(\lambda f, E) = |\lambda| S(f,E)$

(III). $\exists\, C < \infty .\ \forall f,g \in E$: $\cancel{S(f+g, E)}$

$$S(f+g, E) \leq C \left[S(f,E) + S(g,E) \right].$$

Rmk: S is mimicing an average.

Ex 1: $X = \mathbb{R}^n$, $\mathbb{E} := \{\text{dyadic cubes}\}$ (or balls).

$\sigma(Q) = 2^{nk}$ if $l(Q) = 2^k$.

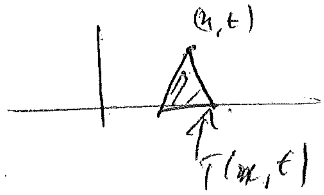
$S(f,Q) := \frac{1}{|Q|} \int_Q |f|\, dx$. or more generally,

$S_p(f,Q) := \left(\frac{1}{|Q|} \int_Q |f|^p\, dx \right)^{1/p}$.

Here, μ generates a measure (Lebesgue).

Ex 2: (Carleson example).

$X = \mathbb{R}^2_+ = \mathbb{R} \times (0,\infty)$; $\mathbb{E} = \{$ set of tents. $T(x,t)$. $\}$.



$\sigma(T(x,t)) := t$.

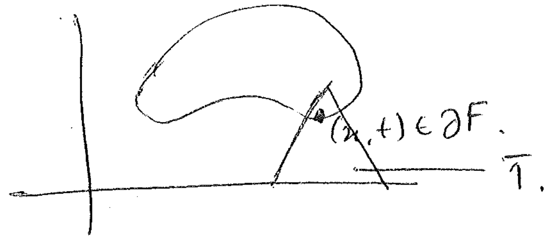
$S(f,T(x,t)) := \frac{1}{t} \iint |f| \frac{dx\,dt}{t}$, or more generally..

$S_p(f,T(x,t)) = \left(\frac{1}{t} \iint |f|^p \frac{dx\,dt}{t}\right)^{1/p}$.

Claim: The only measurable sets is $\emptyset$ and $\mathbb{R}^2_+$.

Pf. Assume that $F \subset X$ is μ-measurable, meaning for all $E \in \mathbb{E}$:

$$\mu(E) = \mu(E \cap F) + \mu(E \cap F^c).$$



Pick T with (x,t). near the top.

$\mu(T) = t.$, $\mu(T \cap F) \gtrsim t$, $\mu(T \cap F^c) \gtrsim t$.

since if $G \ni (x,t) \Rightarrow \mu(G) \geq t$.

So $\cancel{\mu(T) =}$ $\mu(F) \not\geq \mu(T \cap F) + \mu(T \cap F^c)$.

only if $F = \emptyset$, or $F = \mathbb{R}^2_+$.

③ Normed (quasi) derived from σ and S.

A size S gives a notion of a supremum.

$$\cancel{\text{outersup}}\ \text{outsup}_F\, S(f) = \sup_{E\in\mathbb{E}} S(f\,1_F, E).$$

<u>Ex.</u> (Carleson) $\text{outsup}_{\mathbb{R}^2_+} S_1(f) \neq \operatorname{ess\,sup}_{\mathbb{R}^2_+} |f|$.

Let f be sptd on $\mathbb{R}\times(1,\infty)$.

$$\iint_{\mathbb{R}^2_+} |f| \,\frac{dx\,dt}{t} \le C < \infty \quad \text{and unbounded.}$$

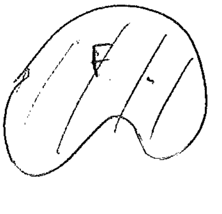
$\operatorname{ess\,sup} |f| = \infty$, but $S_1(f,T) \lesssim 1 \cdot \left(\iint_T |f| \,\frac{dx\,dt}{t}\right) \le C$.

~~The car a crucial~~

The crucial def^n for L^p-spaces.

<u>Def^n 2.5:</u> $f\in B(X)$, $\lambda>0$. Then we define the <u>super level measure</u> as:

$$\mu(S(f)>\lambda) := \inf\{\mu(F) : \text{outsup}_{X\setminus F}\, S(f) \le \lambda\}.$$



$\sim \{x : |f(x)| > \lambda\}$ (Classical).

<u>Recall that</u> $\int |f|^p \,\cancel{dx} = \int \lambda^{p-1}\mu(\{|f(x)|>\lambda\})\,d\lambda$

Use this as motivation to define L^p.

L^p spaces:

$$\|f\|_{L^p(X,\sigma,S)} := \left(\int_0^\infty p\lambda^{p-1} \mu(S(f)>\lambda)\, d\lambda \right) \qquad 0<p<\infty$$

$$\|f\|_{L^\infty(X,\sigma,S)} := \operatorname*{out\,sup}_X S(f).$$

$$\|f\|_{L^{p,\infty}(X,\sigma,S)} := \left(\sup_{\lambda>0} \lambda^p \mu(S(f)>\lambda) \right)^{\frac{1}{p}}.$$

Let $\|\cdot\|_Y$ denote any of these:

(1). $|f| \leq |g| \Rightarrow \|f\|_Y \leq \|g\|_Y$.

(2). $\|\lambda f\|_Y = |\lambda| \, \|f\|_Y$.

(3). $\exists\, C<\infty \ \forall f, g$:

$$\|f+g\|_Y \leq C\left(\|f\|_Y + \|g\|_Y \right).$$

Details: need corresponding estimates for super-level measures.

Want: $\mu(S(f+g) > 2C\lambda) \lesssim \mu(S(f)>\lambda) + \mu(S(g)>\lambda)$.

Let $\varepsilon>0$, take $F_f, F_g \subset X$ s.t.

$\sup_E S(f\cdot 1_{X\setminus F_f}, E) \le \lambda$, $\mu(F_f) \le \mu(S(f)>\lambda)+\varepsilon$.

$\sup_E S(g\cdot 1_{X\setminus F_g}, E) \le \lambda$, $\mu(F_g) \le \mu(S(g)>\lambda)+\varepsilon$.

Let $F = F_f \cup F_g$.

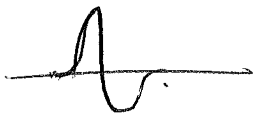
$$\sup_E S((f+g)1_{X\setminus F}, E)$$
$$\le C\left(\sup_E S(f1_{X\setminus F},E) + \sup_E S(g 1_{X\setminus F}, E)\right).$$

$S(f1_{X\setminus F}, E) \le S(f 1_{X\setminus F_f}, E)$ ~~since~~ $F_f \subset F$.

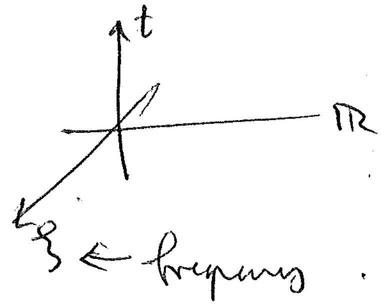
So, $\sup_E S((f+g)1_{X\setminus F}, E) \le 2C\lambda$.

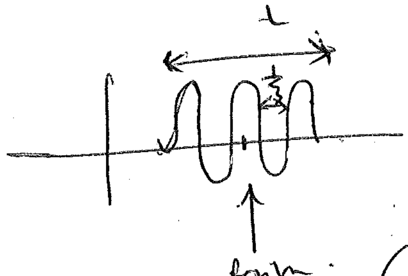
$$\Rightarrow \mu(S(f+g) > 2C\lambda) \le \mu(F) \le \mu(F_f)+\mu(F_g)$$
$$\le \mu(S(f)>\lambda) + \mu(S(g)>\lambda) + 2\varepsilon.$$

Point. (I). $X = \mathbb{R}^2_+$.



(II) $X = \mathbb{R}^3_+$.





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$$\lesssim \sum_{k,i} \int_{E_k^i \setminus F_{k+1}} |g| \frac{du\,dt}{t}$$

$$\lesssim \sum_{k,i} \int_{E_k^i} |g| \mathbb{1}_{F_{k+1}^c} \frac{du\,dt}{t}$$

$$\lesssim \sum_{k,i} S_1\left(g \mathbb{1}_{F_{k+1}^c}, E_k^i\right) \sigma(E_k^i)$$

$$\le \sum_{k,i} 2^k \sigma(E_k^i)$$

$$= \sum_k 2^{k+1} \underbrace{\sum_i \sigma(E_k^i)}_{\lesssim \mu(F_k)}$$

$$\le \sum_k 2^{k+1} \mu(F_k) \simeq \|g\|_{L^1(\mathbb{R}^2_+, \sigma, S_1)}$$